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Abstract. In this article we study, via a kinetic description, the effect of different psychologies on the evolution of the opinion with respect to a binary choice, in a closed group. We show that the interaction between individuals with different reactions regarding the exchange of opinion induces some phenomena, such as the concentration of opinions or the cyclic-in-time behaviour of the distribution function. We provide an existence and uniqueness result for the model and numerically test it in some relevant cases.

1. Introduction

The kinetic approach in sociophysics is a promising line of research to explain collective behaviours in a simple, but mathematically solid, way.

A kinetic model consists in a set of partial integro-differential equations that govern the time evolution of a probability density function, which fully describes the system. The independent variables of the unknown function are the time, and other physical quantities which are relevant to the problem. When dealing with rarefied gases, the typical independent variables are the position and the velocity of the gas particles. In opinion dynamics, a common independent variable is the opinion – or the agreement – with respect to a question.

The introduction of kinetic models in the sociophysical literature started some years ago [17, 16, 18]. This methodology has recently experienced a renewal of attention [19, 1, 6, 7, 9, 11, 5] for many reasons, see also the review article [8].

In this article we study the time evolution of the opinion, with respect to questions of binary type (e.g. a referendum), in a closed community. This problem is a classical issue of sociophysics, and many authors have considered it, see, for example, [10, 15, 19, 9, 6, 7, 5].

In our model, based on a kinetic approach, the unknown is a probability density function that depends on two independent variables: time and opinion of the agent regarding the aforementioned binary question. Each individual of the population can modify his opinion only through the binary exchange of ideas with another member of the community.

Nevertheless, even if we only take into account this elementary phenomenon of opinion exchange, the collective behaviour of the population with respect to the binary question is far from being simple. Indeed, it is well accepted in the literature that many different behaviours concerning the dynamics of opinion formation depend on the fact that the way people think is not uniform. A realistic model should therefore include as many binary interaction rules between individuals as mental paths inside the population.

For simplicity reasons, we only consider here two ways of thinking. As we shall show, even this simplified situation leads to interesting phenomena.

The first psychological attitude is typical of individuals which tend to compromise after an opinion exchange. This behaviour is widely recognized, and it can be considered as the most common one [2, 3, 15, 10]. Besides, it is also the main common feature of the kinetic approaches, as it is emphasized in [8].

The other mental path is completely different, and leads to psychological dynamics of contradictory type. This behaviour has been proposed by Galam in [13, 14], and is based on the fact that some people are deliberately opposed to the choice of the interlocutors, whatever that choice may be.

These two psychological behaviours are translated in our paper by different kinds of collisional rules of kinetic type, which permit to obtain, in a deterministic way, the post-interaction opinions starting from the pre-collisional ones.

It is worth noting that the collision rules proposed here are only examples of possible behaviours. Actually, the psychology of an individual cannot be considered as a mechanical system and it may vary interaction by interaction. It is moreover clear that, in a real situation, the phenomenology is much more intricate (the effect of mass media can be decisive, for example [5]). Nevertheless, as we shall see, the presence of only two fixed behaviours in the context of interpersonal communication is already enough to explain many interesting phenomena, such as the concentration towards some particular opinions or the cyclic (in time) behaviour of the distribution function, two phenomenologies that have been sociologically observed [14].

In our model, we suppose that the population is closed. It means that the total number of individuals is constant. This assumption does not imply a great limitation, since the characteristic times of opinion evolution are very small with respect to typical characteristic times in population dynamics. Moreover, we suppose that the probability of a binary interaction is constant. In a structured society, this hypothesis is not true. Indeed, people are normally involved in a social network, and hence some meetings are much more probable than other ones. However, the influence of a network cannot be explained as a function of opinion and time. Since we restrain ourselves to consider only those two independent variables, it seems logical to treat, as a first approximation, the probability of a binary interaction as a constant. We finally note that our description is well adapted only when the size of the population is large enough: this is the key-point of any kinetic model.

All the previous assumptions have a double effect. On the one hand, they reduce the applicability of the model, on the other hand, by simplifying the phenomena taken into account, they allow to build a model which is tractable from a mathematical point of view.

The organization of the paper is the following. We first describe our model, and put in light different collision rules for both groups of agents. Then we prove an existence result for the considered problem. Eventually, we present some relevant numerical tests and provide an analysis of the quantitative results from a sociological point of view.

2. The kinetic model

In this paper, we study the opinions regarding a binary question (e.g. a referendum) in a non-homogeneous population composed by individuals with different psychologies. We aim to estimate how different possible reactions with respect to the opinion exchange can influence its global behaviour.

In the following, Ω denotes the open interval $(-1, 1)$. We describe the opinion by means of a continuous variable $x \in \bar{\Omega}$, where $x = -1$ and $x = 1$ are identified with the two extreme positions. Any intermediate value between those values means that the corresponding individual partially agrees with the opinion labelled with the same sign, with a degree of conviction which is proportional to $|x|$. If $x = 0$, the corresponding individual has no preference with respect to the question.

We describe the population by using a kinetic approach. The main tool of the model is the concept of *distribution function*, a quantity which depends on time t and opinion variable x, and on the features of the population. Its time evolution, governed by a partial differential equation, then allows to forecast the behaviour of the system. Since the population is non-homogeneous, a possible strategy of description consists in stratifying the individuals with respect to their psychological reactions during the opinion exchange process.

We assume that the members of the population belong to two different groups, and introduce two associated distribution functions which separately describe each group: the population can be split into *conciliatory* people and *contradictory* ones (the precise definitions is given later on). This assumption is the simplest one that allows to investigate the effects of different psychologies on the global behaviour of the population. A generalization of the kinetic approach to more complex situations is obviously possible. In that case, the distribution function is of vectorial type, with as many entries as the possible psychological behaviours inside the population. Note that we do not pay attention to the spatial structure of the closed community, which is assumed interlinked. The two groups are respectively described by the density functions $f = f(t, x)$ and $g = g(t, x)$. Both are defined on $\mathbb{R}_+ \times \bar{\Omega}$.

Once individuated the population to study, if the opinions are defined on a subdomain D of $\bar{\Omega}$, the integrals

$$\int_D f(t, x)\, \mathrm{d}x \qquad \text{and} \qquad \int_D g(t, x)\, \mathrm{d}x$$

represent the number of conciliatory and contradictory individuals with opinion included in D at time t. Note that, in order to give a sense to the previous considerations, we must suppose that f and g satisfy $f(t,\cdot) \in L^1(\Omega)$ and $g(t,\cdot) \in L^1(\Omega)$ for all $t \in \mathbb{R}^+$.

As sketched in the introduction, we only take into account one process of opinion formation given by the interaction between agents, who exchange their point of view and influence themselves. Moreover, we suppose that the interactions between individuals are only of binary type. Multiple interactions can be seen as the result of a chain of binary exchanges.

We model this binary process by borrowing the collisional mechanism of a typical interaction in the kinetic theory of gases: whereas in rarefied gas dynamics the particles exchange momentum and energy in such a way that the principles of classical mechanics are satisfied, here the collisions between individuals allow the exchange of opinions. Since there are two categories of people within the population, we define three types of interactions. We assume that the collision mechanisms do not destroy the bounds of the interval $\bar{\Omega}$. We shall present the collision rules in Section 4.

Each post-collisional opinion can be written as a function of the pre-collisional opinions and depends on the psychologies of the individuals. The interactions between individuals are described by a collisional integral of Boltzmann type, which has the classical structure of a dissipative Boltzmann kernel. Each collisional integral can be viewed as composed of two parts: a *gain term*, which quantifies the exchanges of opinion between individuals which give, after the interaction with an other individual, the opinion x, and a *loss term*, which quantifies the exchanges of opinion where an individual with pre-collisional opinion x experiences an interaction with another member of the population.

It is apparent, in general, that the existence of a pre-collisional pair, which returns a given post-collisional pair after interaction, is not guaranteed, unless we suppose that the collisional rule is a diffeomorphism of $\bar{\Omega}^2$ onto itself. Unfortunately, this assumption is not easy to satisfy. In order to overcome this difficulty, we use a weak form of our problem, in the variable x only, which seems a natural framework for such collision rules, as in [6].

The weak form of the collision kernels are presented below, where $\varphi = \varphi(x) \in C^0(\bar{\Omega})$ is a test function.

Conciliatory-conciliatory interactions. Let x, $x_* \in \bar{\Omega}$ the pre-collisional opinions of two conciliatory agents, and $x^Q, x_*^Q \in \bar{\Omega}$ the opinions after the interaction.

We denote by $Q(f,f)$ the associated kernel, which is defined by

$$\langle Q(f,f),\varphi\rangle = \beta_Q \iint_{\Omega^2} f(t,x)f(t,x_*) \left[\varphi\left(x^Q\right) - \varphi(x)\right] \,\mathrm{d}x\,\mathrm{d}x_*, \tag{1}$$

and $Q^+(f,f)$ the gain part of $Q(f,f)$, namely

$$\langle Q^+(f,f),\varphi\rangle = \beta_Q \iint_{\Omega^2} f(t,x)f(t,x_*)\varphi\left(x^Q\right)\, \mathrm{d}x\, \mathrm{d}x_*.$$

Conciliatory-contradictory interactions. Let x, $x_* \in \bar{\Omega}$ the respective pre-collisional opinions of a conciliatory agent and a contradictory one before an interaction, and $x^R, x_*^R \in \bar{\Omega}$ the opinions after collision.

We denote by $R_1(f,g)$ and $R_2(f,g)$ the associated kernels, which respectively contribute to the time evolution of f and of g. They are defined by

$$\langle R_1(f,g),\varphi\rangle = \beta_R \iint_{\Omega^2} f(t,x)g(t,x_*)\left[\varphi\left(x^R\right)-\varphi(x)\right]\, \mathrm{d}x\, \mathrm{d}x_*, \tag{2}$$

$$\langle R_2(f,g),\varphi\rangle = \beta_R \iint_{\Omega^2} f(t,x)g(t,x_*)\left[\varphi\left(x_*^R\right)-\varphi(x)\right]\, \mathrm{d}x\, \mathrm{d}x_*. \tag{3}$$

Their gain parts are respectively $R_1^+(f,g)$ and $R_2^+(f,g)$, i.e.

$$\begin{aligned}\langle R_1^+(f,g),\varphi\rangle &= \beta_R \iint_{\Omega^2} f(t,x)g(t,x_*)\varphi\left(x^R\right)\, \mathrm{d}x\, \mathrm{d}x_*,\\ \langle R_2^+(f,g),\varphi\rangle &= \beta_R \iint_{\Omega^2} f(t,x)g(t,x_*)\varphi\left(x_*^R\right)\, \mathrm{d}x\, \mathrm{d}x_*.\end{aligned}$$

Contradictory-contradictory interactions. Let x, $x_* \in \bar{\Omega}$ the pre-collisional opinions of two contradictory individuals, and $x^S, x_*^S \in \bar{\Omega}$ the post-collisional ones.

We eventually define the associated kernel $S(g,g)$ by

$$\langle S(g,g),\varphi\rangle = \beta_S \iint_{\Omega^2} g(t,x)g(t,x_*)\left[\varphi\left(x_*^S\right)-\varphi(x)\right]\, \mathrm{d}x\, \mathrm{d}x_*. \tag{4}$$

The associated gain term is denoted by $S^+(g,g)$, and defined by

$$\langle S^+(g,g),\varphi\rangle = \beta_S \iint_{\Omega^2} g(t,x)g(t,x_*)\varphi\left(x_*^S\right)\, \mathrm{d}x\, \mathrm{d}x_*.$$

Analytical form of the model. In all collisional terms, the parameters β_Q, β_R and β_S govern the probability that the associated interaction can occur. In our model, we choose them as constants. This is the simplest possible assumption, which means that the probability of interaction of two individuals does not depend on their respective opinions. Of course, other choices, based on sociological considerations, are possible.

Note that, in (1)–(4), the post-collisional opinions x^Q, x_*^Q, x^R, x_*^R, x^S and x_*^S only appear as an argument of the test-function φ. It is also clear that the collision operators only act on the opinion variable. Moreover, if we suppose that $f(t,\cdot)$ and $g(t,\cdot)$ lie in $L^1(\Omega)$, then the gain and loss parts of the operators, and consequently, the operators themselves, also lie in $L^1(\Omega)$ for any t.

Let $T > 0$. The evolution of f and g is given by the following system of integro-differential equations, in the weak sense in the variable x, where both f^{in} and g^{in} are $L^1(\Omega)$ and nonnegative,

$$\partial_t f = Q(f,f) + R_1(f,g), \tag{5}$$

$$\partial_t g = R_2(f,g) + S(g,g), \tag{6}$$

with the initial conditions

$$f(0,x) = f^{\text{in}}(x), \qquad g(0,x) = g^{\text{in}}(x) \qquad \text{a.e. } x \in \Omega. \tag{7}$$

Equations (5)–(6) take sense for almost every $t \in [0,T]$, for test-functions $\varphi \in C^0(\bar{\Omega})$.

3. Mathematical properties

This section is devoted to state and study some mathematical properties of (5)–(7), which do not depend on the collision rules. We first obtain some a priori estimates and then deduce a result which ensures the existence of weak solutions to (5)–(7).

Proposition 1. *Let (f,g) be a nonnegative weak solution to* (5)–(7) *with nonnegative initial data f^{in}, $g^{\text{in}} \in L^1(\Omega)$. Their respective total masses are conserved, i.e. for almost every $t \in [0,T]$,*

$$\|f(t,\cdot)\|_{L^1(\Omega)} = \left\|f^{\text{in}}\right\|_{L^1(\Omega)}, \qquad \|g(t,\cdot)\|_{L^1(\Omega)} = \left\|g^{\text{in}}\right\|_{L^1(\Omega)}.$$

Proof. We take $\varphi = 1$ as the test function in (5)–(6). □

Proposition 1 means that there is conservation of both species of the population. This property is not realistic if we consider long-time forecasts. Indeed, in such situations, we should also consider processes of birth and death, which also lead to oscillations in the total number of individuals. But usually, as in the case of elections or referendums, the interest of such models is to deduce short-term forecast by using, as an initial datum, poll results. The quantities of interest are then the *macroscopic observables*

$$\int_{\Omega_-} f(t,x)\,\mathrm{d}x, \ \int_{\Omega_+} f(t,x)\,\mathrm{d}x, \ \int_{\Omega_-} g(t,x)\,\mathrm{d}x, \ \int_{\Omega_+} g(t,x)\,\mathrm{d}x,$$

where $\Omega_- = (-1,0)$ and $\Omega_+ = (0,1)$. In fact, we shall numerically study those integrals in Section 5.

Since $|x| \le 1$, from the mass conservation, we immediately deduce that all the moments of both f and g are bounded.

Corollary 1. *Let (f,g) be a nonnegative weak solution to* (5)–(7) *with nonnegative initial data f^{in}, $g^{\text{in}} \in L^1(\Omega)$. Then we have, for almost every $t \in [0,T]$,*

$$\left|\int_\Omega x^n f(t,x)\,\mathrm{d}x\right| \le \left\|f^{\text{in}}\right\|_{L^1(\Omega)}, \qquad \left|\int_\Omega x^n g(t,x)\,\mathrm{d}x\right| \le \left\|g^{\text{in}}\right\|_{L^1(\Omega)}.$$

In order to prove the existence of weak solutions to (5)–(7), we first need the following proposition.

Proposition 2. *Let μ_1, μ_2 be nonnegative constants, σ_1, σ_2 nonnegative functions in $C^0([0,T];L^1(\Omega))$, and u^{in}, v^{in} nonnegative initial data in $L^1(\Omega)$. The system*

$$\partial_t u + \mu_1\, u = \sigma_1, \qquad \partial_t v + \mu_2\, v = \sigma_2, \tag{8}$$

with initial conditions

$$u(0,\cdot) = u^{\mathrm{in}}, \qquad v(0,\cdot) = v^{\mathrm{in}}, \tag{9}$$

has a unique solution $(u,v) \in \big(C^0([0,T];L^1(\Omega))\big)^2$. Moreover, both u and v are nonnegative.

Proof. Let us note that (8)–(9) is a linear ordinary differential system. The Cauchy-Lipschitz theorem implies the existence and uniqueness of the solution to (8)–(9). By using Duhamel's formula, we also know that this solution satisfies, for any t and for a.e. x,

$$\begin{aligned} u(t,x) &= u^{\mathrm{in}}(x)\, e^{-\mu_1 t} + \int_0^t e^{\mu_1(s-t)}\sigma_1(s,x)\,\mathrm{d}s,\\ v(t,x) &= v^{\mathrm{in}}(x)\, e^{-\mu_2 t} + \int_0^t e^{\mu_2(s-t)}\sigma_2(s,x)\,\mathrm{d}s. \end{aligned}$$

The explicit form of the solution ensures the nonnegativity and time continuity of u and v with respect to t. □

Thanks to the previous result, we can now prove the following existence theorem, whose proof is quite similar to the one of the main result in [6].

Theorem 1. *Let f^{in}, g^{in} be nonnegative functions in $L^1(\Omega)$. Then there exists $(f,g) \in L^\infty(0,T;L^1(\Omega)) \times L^\infty(0,T;L^1(\Omega))$ which solves (5)–(6) with initial conditions (7), where the equations take sense in $\mathcal{D}'(-T,T)$.*

Proof. Let us set

$$\varrho_f = \int_\Omega f^{\mathrm{in}}(x_*)\,\mathrm{d}x_* \geq 0, \qquad \varrho_g = \int_\Omega g^{\mathrm{in}}(x)\,\mathrm{d}x \geq 0.$$

We consider the sequence $(f_n, g_n)_{n\in\mathbb{N}}$ inductively defined by $f^0 = 0$, $g^0 = 0$, and, for $n \geq 1$, as weak solutions of

$$\partial_t f^{n+1} + (\beta_Q\varrho_f + \beta_R\varrho_g) f^{n+1} = Q^+(f^n,f^n) + R_1^+(f^n,g^n), \tag{10}$$

$$\partial_t g^{n+1} + (\beta_R\varrho_f + \beta_S\varrho_g) g^{n+1} = R_2^+(f^n,g^n) + S^+(g^n,g^n), \tag{11}$$

altogether with the initial conditions $f^n(0,\cdot) = f^{\mathrm{in}}$ and $g^n(0,\cdot) = g^{\mathrm{in}}$.

With a constant test function equal to 1, it is clear that, for all $n \in \mathbb{N}$, we have

$$\int_\Omega f^n\,\mathrm{d}x \leq \varrho_f, \qquad \int_\Omega g^n\,\mathrm{d}x \leq \varrho_g.$$

The existence of f^n and g^n in $C^0([0,T];L^1(\Omega))$ as nonnegative solutions of (10)–(11) is obtained by induction thanks to Proposition 2, remembering that $Q^+(f^n,f^n)$, ${R_1}^+(f^n,g^n)$, ${R_2}^+(f^n,g^n)$ and $S^+(g^n,g^n)$ all belong to $C^0([0,T];L^1(\Omega))$.

We next prove, by induction again, that (f^n) and (g^n) are non decreasing sequences. It is first clear that $f^1 \geq 0 = f^0$ and $g^1 \geq 0 = g^0$. Then we suppose that $f^n \geq f^{n-1}$ and $g^n \geq g^{n-1}$. We can write, in a weak sense,

$$\partial_t(g^{n+1}-g^n)+(\beta_R\varrho_f+\beta_S\varrho_g)(g^{n+1}-g^n)=P(f^n,f^{n-1},g^n,g^{n-1}),$$

where

$$P=R_2^+(f^n,g^n)+S^+(g^n,g^n)-[R_2^+(f^{n-1},g^{n-1})+S^+(g^{n-1},g^{n-1})].$$

It is clear that P lies in $C^0([0,T];L^1(\Omega))$ and is nonnegative. Again, Proposition 2 ensures that $g^{n+1}-g^n$ is a nonnegative solution of the previous equation. In the same way, we also obtain that $f^{n+1} \geq f^n$.

Therefore, by monotone convergence, there exist f, $g \in L^\infty(0,T;L^1(\Omega))$, such that (f^n) and (g^n) converge to f and g, almost everywhere and in $L^\infty(0,T;L^1(\Omega))$.

We still have to prove that (f,g) satisfies the initial conditions (7), which is quite clear, and solves (5)–(6) in a distributional sense. Let us choose a test function $\varphi \in C^0(\bar{\Omega})$, and a test function in $\mathcal{D}([-T,T])$. Equations (5)–(6) can be written under a weak form, using these test functions. We investigate what happens when $n \to +\infty$ in this formulation.

First, the time derivatives and initial data do not induce any difficulty, when $n \to +\infty$. To treat the linear term with the indices $n+1$, we only have to use Proposition 1 to obtain the loss terms of the collision kernels. Eventually, we have to deal with the nonlinear terms involving f^n and g^n. For instance, if we set $M=\sup\{|\varphi(x)|;\ x \in \bar{\Omega}\}$, we can note that

$$\begin{aligned}\iint_{\Omega^2} &|f^n(t,x)g^n(t,x_*)-f(t,x)g(t,x_*)|\ \left|\varphi(x')\right|\ \mathrm{d}x\ \mathrm{d}x_* \\ &\leq \varrho_g M\,\|f^n(t,\cdot)-f(t,\cdot)\|_{L^1(\Omega)}+\varrho_f M\,\|g^n(t,\cdot)-g(t,\cdot)\|_{L^1(\Omega)}\,,\end{aligned}$$

which goes to 0 when $n \to +\infty$. Using this argument or a similar one, it is easy to recover the gain terms of the collision kernels.

We are then able to let n go to $+\infty$ in the weak formulation of (5)–(6) and obtain the required result. □

4. The collision rules

This section is devoted to describe, in a precise way, the psychological dynamics that we take into account. For each individual, the exchange of opinions with another member of the population is represented, in the model, by a collisional rule that quantifies the modifications in the opinions originated by the exchange itself.

In what follows, $(x, x_*) \in \bar{\Omega}^2$ are the pre-collisional opinions of the agents, whereas $(x', x'_*) \in \bar{\Omega}^2$ represent their opinion after the discussion, where the primes denote the types of variables Q, R and S.

As we shall see, the choice of the collision rules is crucial and heavily conditions the time evolution of the distribution functions given by Equations (5)–(7).

We here propose two models, based on two different psychological mechanisms for contradictory individuals, which allow to describe two types of collective behaviour observed in real situations. Both psychological rules translate the idea that the contradictory way of thinking tends to oppose the effects of the consensus rule.

An essential ingredient of both collision rules is the *attraction function* η, a smooth function which describes the degree of attraction of the average opinion with respect to the starting opinion of the individual. In the first model, we introduce also the *reaction function* α, a smooth function which modulates the reaction of a contradictory individual during the exchange process.

In order to make the models unaffected by the change of label of the two extreme opinions, we shall always assume that both η and α are even. Moreover, we suppose that $\eta : \bar{\Omega} \to \mathbb{R}$ and $\alpha : \bar{\Omega} \to \mathbb{R}$ are C^1, and such that $0 \le \eta < 1$, $0 < \alpha \le 1$. Consequently, the interactions do not destroy the bounds of the interval $\bar{\Omega}$.

In order to translate the idea, well accepted in the literature [6, 19, 12, 10], that extreme opinions are more stable than moderate ones for conciliatory individuals, we suppose that $\eta'(x) \ge 0$ when $x \ge 0$. We also assume that the attraction and reaction functions are such that the Jacobians of the collisional mechanisms are always non zero.

We present two different models, which we respectively name the twist and swing models. This choice of nomenclature will be clear by observing the numerical results of Section 5. We keep the notations defined in 2 for the post-collisional opinions in each case.

4.1. **Twist model.** In the following paragraphs, we detail the involved collision rules.

4.1.1. *Exchange of opinions between two conciliatory individuals.* Let x, $x_* \in \bar{\Omega}$ the opinions of the two conciliatory agents before an interaction. The interaction is described by the rule defined in [6]: the stronger opinions are less attracted towards the average than the weaker ones. The mechanism which restitutes the post-collisional opinions x^Q, x_*^Q is given by

$$x^Q = \frac{x + x_*}{2} + \eta(x)\frac{x - x_*}{2}, \tag{12}$$

$$x_*^Q = \frac{x_* + x}{2} + \eta(x_*)\frac{x_* - x}{2}. \tag{13}$$

4.1.2. *Exchange of opinions between a conciliatory individual and a contradictory one.* Let x, $x_* \in \bar{\Omega}$ the respective opinions of a conciliatory agent and a contradictory one before an interaction. Whereas the conciliatory individual still follows the consensus rule (12), the post-collisional contradictory opinion is computed using the value given by (13), and then somehow taking the opposite value, to model the contradictory effect. The collision rule that individuates the post-collisional opinions x^R, x_*^R writes

$$x^R = \frac{x+x_*}{2} + \eta(x)\frac{x-x_*}{2}, \tag{14}$$

$$x_*^R = -\alpha(x_*)\left[\frac{x_*+x}{2} + \eta(x_*)\frac{x_*-x}{2}\right]. \tag{15}$$

4.1.3. *Exchange of opinions between two contradictory individuals.* Let x, $x_* \in \bar{\Omega}$ the opinions of the two contradictory agents before an interaction. The opinion exchange gives a pair of post-collisional opinion variables x^S, x_*^S, which are given by

$$x^S = -\alpha(x)\left[\frac{x+x_*}{2} + \eta(x)\frac{x-x_*}{2}\right], \tag{16}$$

$$x_*^S = -\alpha(x_*)\left[\frac{x_*+x}{2} + \eta(x_*)\frac{x_*-x}{2}\right]. \tag{17}$$

4.2. **Swing model.** This model substantially differs from the previous one because of the psychological behaviour of contradictory individuals.

4.2.1. *Exchange of opinions between two conciliatory individuals.* The interaction between two conciliatory individuals is still defined by (12)–(13).

4.2.2. *Exchange of opinions between a conciliatory individual and a contradictory one.* Let x, $x_* \in \bar{\Omega}$ the respective opinions of a conciliatory agent and a contradictory one before an interaction. After interaction, the contradictory individual tends to oppose the post-collisional opinion of his peer, i.e. the post-collisional opinions x^R, x_*^R are defined by

$$x^R = \frac{x+x_*}{2} + \eta(x)\frac{x-x_*}{2}, \tag{18}$$

$$x_*^R = \begin{cases} 1 - \dfrac{(1-x^R)(1-x_*)}{(1-x)} & \text{if } x < x_*, \\ x_* & \text{if } x = x_*, \\ \dfrac{(1+x^R)(1+x_*)}{(1+x)} - 1 & \text{if } x > x_*. \end{cases} \tag{19}$$

We note that the post-collisional opinion x_*^R is well defined because the range of the validity of the formulae prevents the denominators from vanishing. The interaction clearly does not destroy the bounds of $\bar{\Omega}$. Finally, the continuity of the mechanism with respect to (x, x_*) is also ensured when $x = x_*$.

4.2.3. *Exchange of opinions between two contradictory individuals.* Let x, $x_* \in \bar{\Omega}$ the pre-collisional opinions of two contradictory agents. The interaction leading to the post-collisional opinions x^S, x_*^S is given by

$$x^S = \begin{cases} \dfrac{x-1}{2} + \eta(x)\dfrac{x+1}{2} & \text{if } x \le x_*, \\ \dfrac{x+1}{2} + \eta(x)\dfrac{x-1}{2} & \text{if } x > x_*, \end{cases} \tag{20}$$

$$x_*^S = \begin{cases} \dfrac{x_*+1}{2} + \eta(x_*)\dfrac{x_*-1}{2} & \text{if } x \le x_*, \\ \dfrac{x_*-1}{2} + \eta(x_*)\dfrac{x_*+1}{2} & \text{if } x > x_*. \end{cases} \tag{21}$$

It can be seen as the opposite of the consensus rule, i.e. the post-collisional opinions are obtained using the consensus rule (12)–(13) with the extreme opinions ± 1.

4.3. **On the collision mechanisms.** The sets of attraction and reaction functions are not empty. Indeed, a possible choice for functions η and α is $\eta(x) = H$ and $\alpha(x) = A$ for $x \in \bar{\Omega}$, where the constants H and A both satisfy $0 < H, A < 1$. We can easily check that, in that case, the Jacobians of all the collisions rules (12)–(21) are defined and nonzero.

5. Numerical tests

In this section, we present some numerical results concerning both models presented above. The computations were performed by using a numerical code written in `C`.

We consider a regular subdivision $(x_0, \cdots, x_N)$ of Ω, with $N \ge 1$. The functions f and g are computed at the center $x_{i+1/2}$ of each interval $[x_i, x_{i+1}]$, $0 \le i \le N-1$, and we choose $N = 1000$.

Since $\|f^{\text{in}}\|_{L^1(\Omega)}$ and $\|g^{\text{in}}\|_{L^1(\Omega)}$ may not have the same order of magnitude, we replace f and g respectively by

$$\tilde{f} := \frac{f}{\|f^{\text{in}}\|_{L^1(\Omega)}}, \qquad \text{and} \qquad \tilde{g} := \frac{g}{\|g^{\text{in}}\|_{L^1(\Omega)}}.$$

Then $\tilde{f}$ and $\tilde{g}$ solve the same kind of equations (5)–(6) as f and g, but β_Q is replaced by $\beta_Q \times \|f^{\text{in}}\|_{L^1(\Omega)}$, β_S by $\beta_S \times \|g^{\text{in}}\|_{L^1(\Omega)}$, β_R by $\beta_{R_1} := \beta_R \times \|g^{\text{in}}\|_{L^1(\Omega)}$ for R_1 and β_R by $\beta_{R_2} := \beta_R \times \|f^{\text{in}}\|_{L^1(\Omega)}$ for R_2. It is then quite clear that both $\|\tilde{f}(t,\cdot)\|_{L^1(\Omega)}$ and $\|\tilde{g}(t,\cdot)\|_{L^1(\Omega)}$ remain constant and equal to 1, for almost every t.

To numerically perform the collisions, we use a slightly modified Bird method [4], as in [6]. Note that our scheme of course prevents the opinions from going out of $\bar{\Omega}$, and conserves the population of each group.

As we already explained, we are mainly interested in the computation of the quantities

$$I^-(f+g) = \int_{-1}^{0} (f(t,x)+g(t,x))\ \mathrm{d}x,$$
$$I^+(f+g) = \int_{0}^{1} (f(t,x)+g(t,x))\ \mathrm{d}x.$$

When normalized to $\|f^{\mathrm{in}} + g^{\mathrm{in}}\|_{L^1(\Omega)}$, they can be seen as the fraction of agents who respectively favour negative and positive opinions.

In each numerical tests, the collision frequencies are set to $\beta_S = 1$, $\beta_R = 2$, $\beta_Q = 5$. That means that we always consider that the interactions involving contradictory people are less frequent that the ones involving conciliatory individuals. Besides, the initial data are chosen such that $\|f^{\mathrm{in}}\|_{L^1(\Omega)} = 1$ and $\|g^{\mathrm{in}}\|_{L^1(\Omega)} = 0.1$, so that conciliatory individuals are majority, but contradictory people are a significant part of the whole population. We also tried smaller values of $\|g^{\mathrm{in}}\|_{L^1(\Omega)}$. The same kind of behaviours shown below are recovered, but the time scales get smaller and the interesting transient effects cannot really be pointed out.

Eventually, we choose, for $x \in \bar{\Omega}$, $\eta(x) = (1+x^2)/4$, and $\alpha(x) = (1+x^2)/2$ when required. It is quite difficult to find a precise sociological meaning for those choices. We refer to [6] for the discussion about the choice of η. Nevertheless, both functions satisfy the required assumptions, and we emphasize that we have numerically checked that all the Jacobians were nonzero.

5.1. **Twist model.**

5.1.1. *Uniform contradictory group within an uncentred conciliatory population.* In this first test, we show the behaviour of the model with an Heaviside-step-like initial datum for conciliatory individuals, and a constant initial datum for contradictory people:

$$f^{\mathrm{in}}(x) = \begin{cases} 2 & \text{if } x < -0.5, \\ 0 & \text{if } x > -0.5, \end{cases} \qquad g^{\mathrm{in}}(x) = 0.05. \tag{22}$$

In Figure 1, we plot both graphs of distribution functions at an arbitrary time $t = 20$, and we note that f and g are already close to their equilibrium state. As expected, the initial conditions for both groups do not have any influence after a transient time. The distribution functions f and g asymptotically get a Dirac-mass shape. Figure 2 ensures that the Dirac mass centres for both f and g go to 0, since $I^+(f+g)$ goes to $\|f^{\mathrm{in}} + g^{\mathrm{in}}\|_{L^1(\Omega)}/2 = 0.55$. Of course, convergence to the equilibrium for g is slower than the one for f.

The twist model then clearly results in a centred population with no precise opinion, since both Dirac masses are asymptotically centred at 0. It is interesting to note that, in the contradictory-free diffusionless case described in [6], when g does not appear, f also converges to a Dirac mass, but its centre is given by the average opinion of the initial conciliatory population.

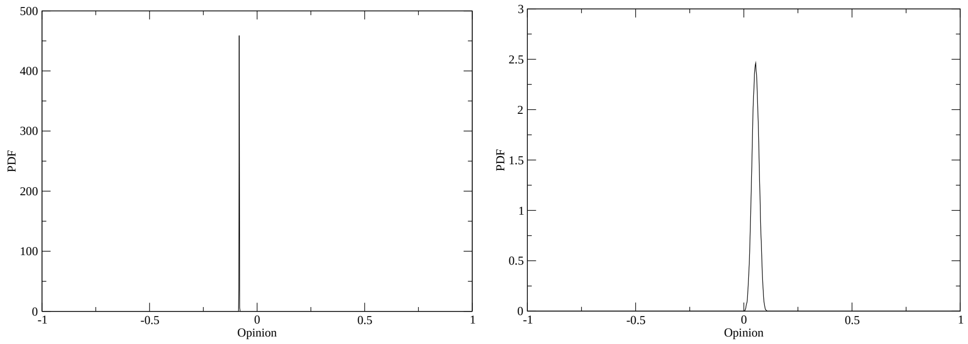

FIGURE 1. Twist model: graphs of (a) f and (b) g at $t = 20$ with initial data (22).

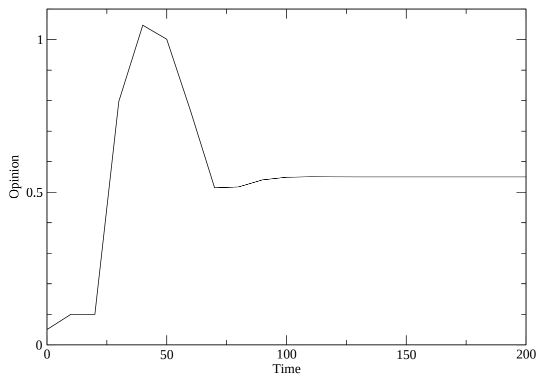

FIGURE 2. Twist model: graph of $I^+(f + g)$ w.r.t. t with initial data (22).

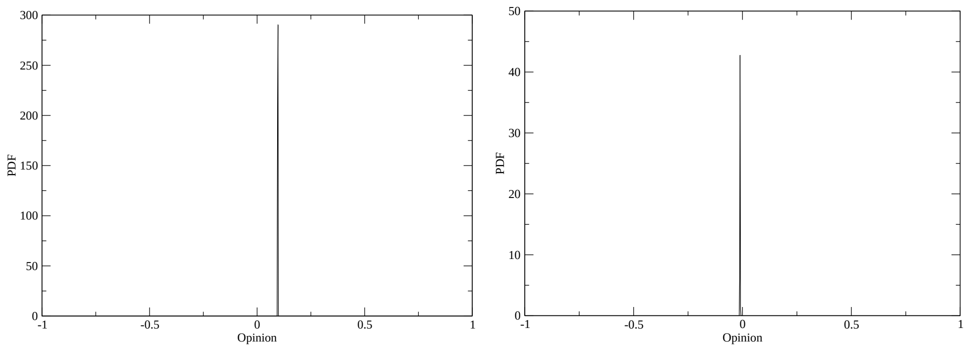

FIGURE 3. Twist model: graphs of (a) f and (b) g at $t = 20$ with initial data (23).

5.1.2. *Disjoint-supported initial data.* Let us now investigate the case when the support sets of the initial data are disjoint. Indeed, in that situation, the support sets of f and g may still remain disjoint because of the collision rules involving contradictory people. More precisely, we consider the following initial data:

$$f^{\mathrm{in}}(x) = \begin{cases} 2 & \text{if } x < -0.5, \\ 0 & \text{if } x > -0.5, \end{cases} \qquad \text{and} \qquad g^{\mathrm{in}}(x) = \begin{cases} 0 & \text{if } x < 0.5, \\ 0.2 & \text{if } x > 0.5. \end{cases} \tag{23}$$

Again, both functions f and g fastly converge towards Dirac masses centred at 0, and the conclusion we obtained in 5.1.1 seems to hold.

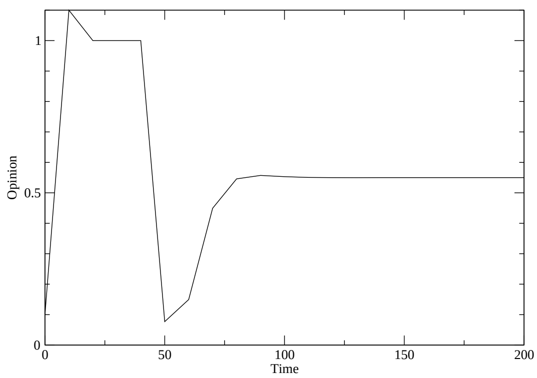

FIGURE 4. Twist model: graph of $I^+(f+g)$ w.r.t. t with initial data (23).

However, for small times, such as $t = 20$, each conciliatory individual has a positive opinion and each contradictory individual a negative one (see Figure 3), a situation which is significantly different with respect to the initial data. After a transient period, where a double reversal of the majority takes place, the system reach an equilibrium configuration. In fact, this effect could be found in 5.1.1 too, see Figure 2.

As a conclusion, the twist model forecasts situations which always lead to a fifty/fifty result. The phenomenon of "hung elections" [13] is the natural issue of this model. But one of the main interest of our model is its behaviour during a transient time, with unexpected results due to contradictory individuals.

5.2. **Swing model.**

5.2.1. *Uniform contradictory group within an uncentred conciliatory population.* We start with the first same numerical test as in 5.1.1, i.e. initial data given by (22).

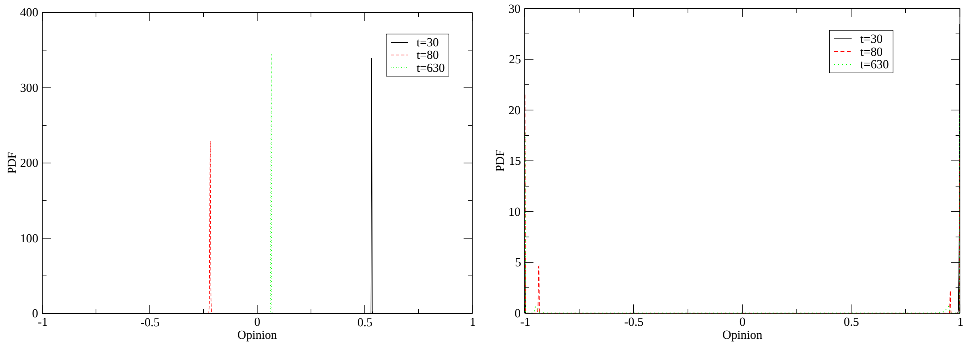

FIGURE 5. Swing model: graphs of (a) f and (b) g, at various times, with initial data (22).

As we can see in Figure 5, the distribution function f is a Dirac mass again, but, this time, its centre seems to randomly swing. Thus, at $t = 30$, each conciliatory individual has a positive opinion, and at $t = 90$, each one of them has a negative opinion. This behaviour is of course linked to the one of g, which seems to converge towards the sum of two Dirac masses, respectively centred at -1 and 1. But in fact, the mass of each Dirac function does not remain constant with respect to t.

Those behaviours are very far from the results given by the twist model. Figure 6 shows that this time, we may not get a fifty/fifty situation, and no tendency points out: when time grows, $I^+(f+g)$ can randomly go below or above $\|f^{\text{in}} + g^{\text{in}}\|_{L^1(\Omega)}/2 = 0.55$, and there is no asymptotic equilibrium.

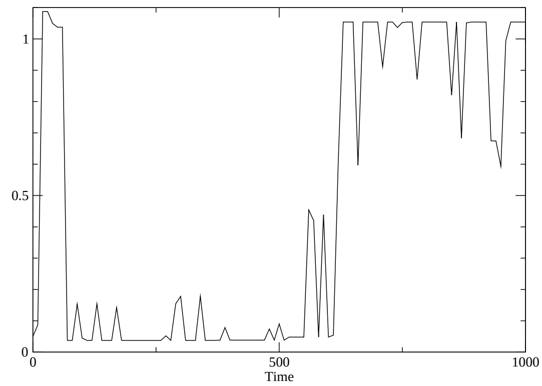

FIGURE 6. Swing model: graph of $I^+(f+g)$ w.r.t. t with initial data (22).

5.2.2. *Disjoint-supported initial data.* Let us now study the behaviour of the swing model with the set of initial data (23).

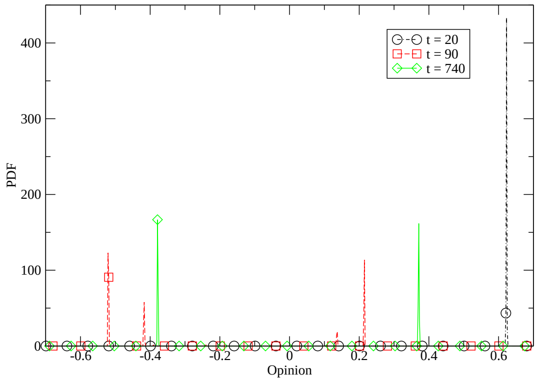

FIGURE 7. Swing model: graph of f, at various times, with initial datum (23).

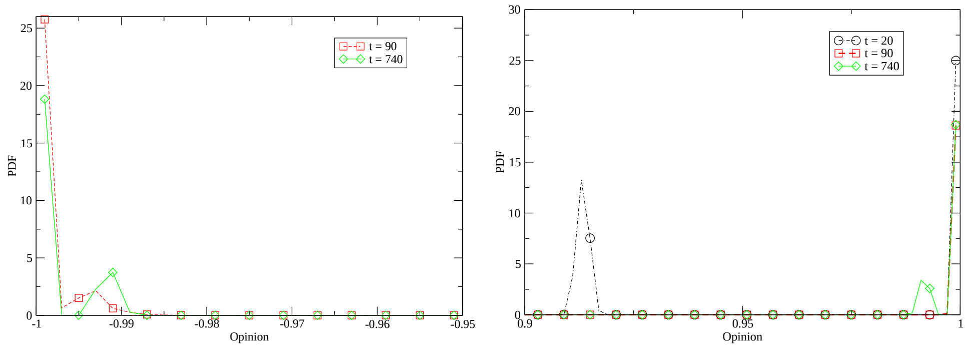

FIGURE 8. Swing model: graph of g near (a) -1 and (b) $+1$, at various times, with initial datum (23).

In Figure 7, we can see that f first concentrates near 0.6, and then have a swinging behaviour, with variable opinion clusters with respect to time. The graph of g is only studied near ± 1: indeed, $g = 0$ outside the domains which are shown on Figure 8. The contradictory distribution function g seems to converge towards a sum of two Dirac masses centred at ± 1, but their respective masses do not remain constant.

Eventually, in Figure 9, we note that there are only two changes of majority in the opinion, and they both happen at small times ($t < 200$). Beyond that time, there is no majority change anymore: even if the majority rate itself is modified, it stays below 0.55. That seems to be significantly different from the case when contradictory people are initially uniformly distributed, where majority changes can happen at any time. Nevertheless, those majority changes are not very frequent anyway, so when we perform a computation with higher final times, in fact, we can recover this possibility of majority change. It ascertains the idea that there is no asymptotic equilibrium at all, and that there exist psychological behaviours which can lead to unpredictable election resultas, where the majority can quickly change.

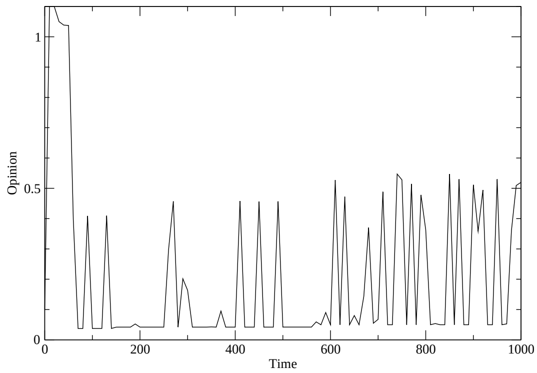

FIGURE 9. Swing model: graph of $I^+(f+g)$ w.r.t. t with initial data (23).

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